Determinacy of Linear Rational Expectations Models*

John H. Boyd III† and Michael Dotsey‡

June 1996

Abstract. In this paper we analyze issues concerning nominal determinacy when the monetary authority uses the interest rate as either an instrument or intermediate target. Analysis of this issue requires the development of a more general framework for investigating the properties of linear rational expectations models. With this framework we are able to show the viability of certain classes of interest rate pegs.

†Department of Economics, University of Rochester, Rochester, NY 14627.

‡Research Department, Federal Reserve Bank of Richmond, P.O. Box 27622, Richmond, VA 23261

*We would like to thank Mary Finn, Marvin Goodfriend, Bennett McCallum, Alan Stockman and the anonymous referees for their helpful comments and suggestions. We are especially indebted to Robert King for many helpful suggestions on numerous versions of this paper.

1. Introduction

This paper provides a general analysis to an interesting and relevant class of models—linear rational expectations models incorporating monetary policies that respond to interest rates. Since this is how the Federal Reserve behaves, we need a systematic way of analyzing economic models that correctly model Fed behavior. In order to accomplish this we develop a fairly general mathematical framework that should be useful in areas other than the particular application addressed in this paper.

In order to accomplish this task we extend the solution procedures for linear rational expectations models developed in Evans and Honkapohja (1986) and Evans (1987) that characterize the possible infinity of general ARMA solutions. We then provide a theorem that allows us to establish the uniqueness and hence determinacy of the solution. All this is covered in the Appendix. This framework is very general and should be useful to researchers interested in incorporating more realistic models of monetary policy into dynamic stochastic models.

Section Two sets up the basic model. In Section Three we use our machinery to analyze selected interest rate rules and show that some of them are not well specified—they produce nominal indeterminacies. Section Four carefully examines policies that peg the nominal interest rate. A short section concludes the paper.

2. Solutions to a General Linear Rational Expectations Model

In this section we characterize the possible infinity of solutions to a general linear rational expectations model.[1] Since none of the exogenous variables in our models follow explo-

[1] In what follows we draw heavily on the work of Blanchard and Kahn (1980), Evans and Honkapohja (1986) and Evans (1987). The general model is similar to that examined in Evans (1987). Our presentation of the characterization follows Evans and Honkapohja (1986), which was better adapted to our situation. The uniqueness results use the methods of Blanchard and Kahn (1980). The approach of Broze, Gourieroux

sive paths, we confine our attention to nonexplosive solutions. Specifically, we confine our analysis to solutions that are polynomially bounded.[2] After characterizing the solutions we present a theorem that determines if a unique solution exists.

2.1. The General Model and Uniqueness

The models of monetary policy consider in this paper all lie within the class of general linear rational expectations models given by

$$0 = \sum_{r=0}^{n} \sum_{s=1}^{k+r} A_{r,s-r} E_{t-r} x_{t+s-r} + \sum_{s=0}^{l} D_s x_{t-s} + \sum_{s=0}^{m} F_s v_{t-s}. \tag{2.1}$$

where x_t is a q-vector, v_t is a q-vector of exogenous random variables with $E_{t-1} v_t = 0$, $D_0 = -I$, $A_{r,s}$ are $q \times q$ matrices with $A_{r,s} = 0$ for $r + s < 1$, D_s and F_s are $q \times q$ matrices and the information set contains all variables date t and earlier.[3] Moreover, D_l, F_m and some $A_{n,s}$ and $A_{r,k}$ are non-zero. That is, we can not write equation (2.1) with a smaller k, l, m or n. In the Appendix we provide the general solution to models of this type. These solutions are characterized by ARMA representations as in Evans and Honkapohja (1986) and Evans (1987) and involve a set of arbitrary MA coefficients. Thus the general solution is not unique. If the model does have a unique solution, finding that solution necessarily involves factoring the general solution until the arbitrary MA coefficients are removed.

Because uniqueness is an important and somewhat subtle issue for this class of models, we develop a theorem that describes the necessary and sufficient condition for unique solutions to (2.1). These conditions involve the relationship between eigenvalues and initial conditions as well as certain rank conditions.

and Szafarz (1985) is not suitable for use here because our solutions are typically not stationary, and may even involve unit roots.

[2] Any solution x_t satisfies $|E_t x_{t+i}| \le (1+i)^{n_t} \bar{x}_t$ for all i where n_t is an integer and $\bar{x}_t$ is a stationary random variable. In models derived from optimizing behavior, the transversality condition will typically rule out polynomially unbounded solutions. Note that unit roots are permitted.

[3] The use of finitely many lagged disturbance terms is not particularly restrictive. If the disturbances were instead generated by an ARMA process, it could be easily converted to the form (2.1), with higher values of l, m, and n.

Let these initial conditions be represented by $QX_0 = N$, where N is $\bar{q} \times 1$, $X_0 = (x_0, E_0x_1, \ldots, E_0x_{t+n+k-1})'$ is $q(n+k) \times 1$, and Q is $\bar{q} \times (n+k)q$. If we update (2.1) n periods (and for simplicity let $l = n$) and take expectations as of time t, the homogeneous part of our system can be written as

$$E_tX_{t+1} = KX_t \tag{2.2}$$

where K is $q(n+k) \times q(n+k)$. Under fairly general conditions K is well-defined and in rational canonical form. The Jordan decomposition of K is given by $K = SJS^{-1}$.

Uniqueness of the solution to (2.1) will depend on a comparison of the number of eigenvalues of K that are on or inside the unit circle and the number of initial conditions. Moreover, we must control the stochastic portion of solutions corresponding to the good roots, and match them with the initial conditions. This is accomplished via the following rank conditions. Define the $q \times (n+k)q$ matrices $C_s = [D_s, A_{s,1-s}, \ldots, A_{s,k}, 0, \ldots, 0]$ for $s = 0, \ldots, n-1$.[4] We say the rank condition is satisfied if the matrices $\sum_{r=0}^{s} C_r S^* J_*^{s-r}$ all have full rank for $s = 0, \ldots, n-1$ where S^* consists of the first q_{in} columns of S associated with the q_{in} eigenvalues on or inside the unit circle and J_* is the conformable $q_{in} \times q_{in}$ part of J. Our uniqueness result, proved in the Appendix, is:

Uniqueness Theorem. *Suppose that J is invertible. Consider the polynomially bounded solutions of (2.1) that obey the initial conditions $QX_0 = N$ with N an $\bar{q}$–vector. There are infinitely many such solutions if $\bar{q} < q_{in}$, and there are usually no such solutions if $\bar{q} > q_{in}$. Suppose further that the rank condition is satisfied, QS^* has full rank, and that $q_{in} \leq q$. Then there is a unique such solution if $q_{in} = \bar{q}$.*

Proof of Uniqueness Theorem[5]. Suppose we have two solutions to equation (2.1), x_t

[4] Note that C_{n-1} does not have a zero at the end.

[5] This proof is based on the procedure used by Blanchard and Kahn (1980). Our result differs from theirs by allowing past expectations and by replacing the predetermined/non-predetermined distinction with appropriate initial conditions.

and x'_t. Updating n periods, $\Delta x_t = x'_t - x_t$ satisfies the homogeneous equation

$$0 = \sum_{r=0}^{n} \sum_{s=1}^{k+r} A_{r,s-r} E_{t+n-r} \Delta x_{t+n+s-r} + \sum_{r=0}^{n} D_r \Delta x_{t+n-r}. \tag{A.1}$$

We can rewrite equation (A.1) as

$$\begin{aligned} 0 &= [D_0, A_{01}, \ldots, A_{0k}, 0, \ldots, 0]\Delta X_{t+n} + [D_1, A_{10}, \ldots, A_{1k}, 0, \ldots, 0]\Delta X_{t+n-1} \\ &\quad + \cdots + [D_n, A_{n,1-n}, \ldots, A_{n,k-1}]\Delta X_t + E_t[0, \ldots, 0, A_{nk}]\Delta X_{t+1} = 0 \qquad \text{(A.2)} \\ &= C_0 \Delta X_{t+n} + \ldots, +C_{n-1}\Delta X_{t+1} + [D_n, A_{n,1-n}, \ldots, A_{n,k-1}]\Delta X_t \\ &\quad + E_t[0, \ldots, 0, A_{nk}]\Delta X_{t+1}. \end{aligned}$$

We get a second perspective on equation (A.1) by taking expectations at time t, yielding

$$0 = \sum_{r=0}^{n} \sum_{s=1}^{k+r} A_{r,s-r} E_t \Delta x_{t+n+s-r} + \sum_{r=0}^{n} D_r E_t \Delta x_{t+n-r}.$$

Any solution to equation (A.2) must satisfy this equation also. Substituting, we find $0 = \sum_{r=0}^{n+k} \tilde{B}_r E_t \Delta x_{t+r}$, which we can write in matrix form as $E_t \Delta X_{t+1} = K \Delta X_t$. Pesaran (1987) shows that the general solution is $\Delta X_t = SJ^t \mathbf{M}_t = K^t S \mathbf{M}_t$ where $\mathbf{M}_t$ is an arbitrary martingale ($E_t \mathbf{M}_{t+1} = \mathbf{M}_t$ for all t). We can rewrite this as $S^{-1}\Delta X_t = (\zeta_1 \mathbf{M}_{1t}, \ldots, \zeta_{(n+k)q} \mathbf{M}_{(n+k)q,t})'$ where the ζ_i are the eigenvalues of K. Since ΔX_t (and hence $S^{-1}\Delta X_t$) is polynomially bounded, the $\mathbf{M}_{it}$ corresponding to $|\zeta_i| > 1$ must be zero. As the eigenvalues are ordered by increasing modulus, only the first q_{in} components of $\mathbf{M}_t$ are non-zero. We denote these upper q_{in} components of $\mathbf{M}_t$ by $\mathbf{M}^*_t$.

We first tackle the case $q_{in} \neq \bar{q}$. Consider the initial condition $QS\mathbf{M}_0 = Q\Delta X_0 = 0$. Since only the first q_{in} components of $\mathbf{M}_0$ are non-zero, we may rewrite this as $QS^*\mathbf{M}^*_0 = 0$. If there are fewer independent initial conditions than unknowns ($q_{in} > \bar{q}$), there will be infinitely many deterministic solutions to $QS^*\mathbf{M}_0^* = 0$, and hence to the system (2.1). However, if $q_{in} < \bar{q}$, there will usually not be enough non-zero components of X_0^* to even satisfy the initial conditions, and (2.1) will not have solutions.

Now consider the case $q_{in} = \bar{q} \leq q$. The next step here is to tame the remaining martingale components of $\mathbf{M}_t^*$ by showing they are deterministic. We substitute this solution back in (A.2) to obtain further restrictions on $\mathbf{M}_t$. Note that

$$C_0 S J^{t+n}\mathbf{M}_{t+n} + \cdots + C_{n-1} S J^{t+1}\mathbf{M}_{t+1} + [D_n, A_{n,1-n}, \ldots, A_{n,k-1}] S J^t \mathbf{M}_t \\ + [0, \ldots, 0, A_{nk}] S J^{t+1}\mathbf{M}_t = 0. \quad \text{(A.3)}$$

Since these are the only non-zero components of $\mathbf{M}_t$, this equation reduces to

$$C_0 S^* J_*^{t+n}\mathbf{M}_{t+n}^* + \cdots + C_{n-1} S^* J_*^{t+1}\mathbf{M}_{t+1}^* + [D_n, A_{n,1-n}, \ldots, A_{n,k-1}] S^* J_*^t \mathbf{M}_t^* \\ + [0, \ldots, 0, A_{nk}] S^* J_*^{t+1}\mathbf{M}_t^* = 0. \quad \text{(A.4)}$$

If the first matrix of (A.4) has full rank (i.e., q), we may write $\mathbf{M}_{t+n}^*$ as a linear combination of $\mathbf{M}_t^*, \ldots, \mathbf{M}_{t+n-1}^*$. This only works if $\mathbf{M}_{t+n}^*$ has at most q components. This implies that $\mathbf{M}_{t+n}^*$ is measurable with respect to information at time $t+n-1$. This measurability implies $E_{t+n-1}\mathbf{M}_{t+n}^* = \mathbf{M}_{t+n}^*$. But $E_{t+n-1}\mathbf{M}_{t+n}^* = \mathbf{M}_{t+n-1}^*$ because $\mathbf{M}_t^*$ is a martingale, so $\mathbf{M}_{t+n}^* = \mathbf{M}_{t+n-1}^*$. Since the equation holds for $t = 0, 1, \ldots,$ the martingale must be constant after time $t = n-1$. Provided the rank condition is satisfied, we may proceed by induction to find that $\mathbf{M}_t^*$ is constant for all times t. Since $\mathbf{M}_t^*$ is constant, we have a deterministic difference equation for ΔX_t. Thus ΔX_t is completely determined by the first q_{in} components of ΔX_0 (which are ΔX_0's only non-zero components).

Consider again the initial condition $QS^*\mathbf{M}_0^* = 0$. As usual, solving (2.1) reduces to solving the related homogeneous equation (without the v_t terms). If $q_{in} = \bar{q}$, we will always be able to find X_0^* solving $QS^*X_0^* = N$, and solve the original equation. $\square$

Notice that uniqueness does not depend on the stochastic properties of the shocks and that our eigenvalue conditions are similar to those found in Whiteman (1983). We wish to stress, however, the importance of the rank condition, which will correctly indicate nonuniqueness in some later examples—examples that otherwise satisfy the conditions of our theorem and therefore satisfy all the conditions of Whiteman's theorem. Also, our theorem does not require the separation of variables into predetermined and nonpredetermined variables as in Blanchard and Kahn (1980), but merely relies on the properties of the eigenvalues in relation to the number of initial conditions.

3. Appendix

3.1. Uniqueness Theorem

To prove uniqueness, we again start with the general model of equation (2.1), this time in matrix form. Define $B_s = \sum_{r=0}^{n} A_{rs}$ for $s = 1-n, \ldots, k$ and $B_s = 0$ otherwise. We assume B_k is invertible. We specialize to the case $l \leq n$.[6] If $l < n$, we set $D_s = 0$ for $l < s < n$. Define $\tilde{B}_{n+s} = B_s$ for $s = 1, \ldots, k$ and $\tilde{B}_s = D_{n-s} + B_{s-n}$ for $s = 0, \ldots, n$. We assume $\tilde{B}_{n+k}$ is invertible. Define the $(n+k)q$–vector $X_t = (x_t, E_t x_{t+1}, \ldots, E_t x_{t+n+k-1})'$ and let

$$K = \begin{bmatrix} 0 & I & & \\ & 0 & I & \\ & & \cdots & \\ -\tilde{B}_{n+k}^{-1}\tilde{B}_0 & -\tilde{B}_{n+k}^{-1}\tilde{B}_1 & \ldots & -\tilde{B}_{n+k}^{-1}\tilde{B}_{n+k-1} \end{bmatrix}.$$

As it is in rational canonical form, the matrix K has characteristic polynomial $\det[\tilde{B}_0 + \cdots + \zeta^{n+k}\tilde{B}_{n+k}] = 0$. Let J be the Jordan form with the eigenvalues arranged in increasing absolute value and let S diagonalize K so $SJ = KS$. Let q_{in} be the number of eigenvalues

[6] Other cases could be addressed in a similar fashion, but complicate the formulation. We will comment further on this later.

on or inside the unit circle, counted according to multiplicity. Denote the first q_{in} columns of S by S^* and the $q_{in} \times q_{in}$ upper left hand block of J by J_*.

What if $l > n$?. When $l > n$, a similar procedure can be applied. To see the basic idea, consider the case $l > n = 0$. Redefine X_t as the $(l+k)q$–vector

$$X_t = (x_{t-l}, \ldots, x_t, E_t x_{t+1}, \ldots, E_t x_{t+k-1})'$$

and set $\tilde{B}_s = D_{-s}$ for $s = -l, \ldots, 0$. The matrix K can be handled similarly. Here applying E_t to equation (A.1) merely yields (A.1) back again, so the restriction equation (A.4) is null. This might seem to preclude a similar result, as these restrictions were used to eliminate arbitrary martingale terms. However, a close examination of the equation $\Delta X_t = SJ^t\mathbf{M}_t^*$ reveals that the top row of $SJ^{t+1}\mathbf{M}_{t+1}^*$ (Δx_{t-l+1}) is the second row of $K^tS\mathbf{M}_t^*$. If S^*, the upper left-hand block of S, is invertible, and $q_{in} \leq q$, we can conclude $\mathbf{M}_{t+1}^*$ is a linear combination of the entries of $\mathbf{M}_t^*$. Thus $\mathbf{M}_{t+1}^*$ is measurable at time t and equal to $\mathbf{M}_t^*$ by the martingale property. This means that the martingale terms are actually deterministic, and the rest of the argument of the uniqueness theorem applies.[7]

Constant Terms. The addition of a constant term to equation (2.1) merely adds a linear time trend to the solution when there are no unit roots of the characteristic equation. When there are unit roots, it adds a non-linear trend with the power of t being one greater than the multiplicity of the unit root.

Singular J. The uniqueness theorem could be extended to cover J singular at the cost of complicating the statement. The presence of a zero eigenvalue can sometimes reduce the number of initial conditions required, depending on how lagged values enter the equation.

[7] If $l > 1$, the next pair of rows can also be used. This might allow the requirement that $q_{in} \leq q$ to be relaxed.

Example 1: Consider example C from Blanchard and Kahn, which is not covered by their theorem. It falls into the zero eigenvalue case. In our notation, their example is $x_t = aE_{t-1}x_t + \omega_t$ with $E_{t-1}\omega_t = 0$, and we will assume $a \neq 1$. This is very easy to solve directly. Applying E_{t-1}, we obtain $E_{t-1}x_t = aE_{t-1}x_t$, so $E_{t-1}x_t = 0$. It follows that the unique solution is $x_t = \omega_t$. In this case, $n = q = 1$, $k = m = 0$, $D_0 = -1$ and $A_{10} = a$. Then $\tilde{B}_0 = 0$ and $\tilde{B}_1 = a - 1$ and $K = 0$. The rank condition is satisfied, and one might suppose that one initial condition is needed for uniqueness. However, the fact that $K = 0$ here kills off any possible dependence on initial conditions, and an initial condition would lead to no solution, unless $x_0 = \omega_0$.

3.2. General ARMA solutions

Here we extend Propositions 1 and 2 in Evans and Honkapohja (1986) to our context. These characterize the general form of the ARMA solutions for the models examined in our paper. The proofs basically follow their structure, and will be omitted. Their Propositions 3 and 4 also apply to our work.

Applying E_{t-n} shows that any solution to (2.1) also solves

$$0 = \sum_{s=1-n}^{k} B_s E_{t-n} x_{t+s} + \sum_{s=0}^{l} D_s E_{t-n} x_{t-s} + \sum_{s=n}^{m} F_s v_{t-s}. \tag{A.5}$$

where lagged values of v only appear when $m \geq n$. Our strategy is to look for general ARMA solutions to (A.5), and then substitute back into (2.1) to see which of these solve the original model.

Proposition 1. *(General ARMA form of solution) Suppose $q = 1$, and let x_t be a stochastic process satisfying (A.5) and having a finite ARMA representation of lowest AR degree given by*

$$x_t = \Pi(L)x_{t-1} + \Psi(L)v_t + \chi(L)\omega_t$$

where ω_t is arbitrary white noise that is uncorrelated with v_t. Then $\deg \Pi \leq k + l^* - 1$, $\deg \Psi \leq k + m^*$, *and* $\deg \chi \leq k + n - 1$ *where* $l^* = \max\{l, n-1\}$ *and* $m^* = \max\{m, n-1\}$.

Remark on q. The restriction to $q = 1$ is because, using Evans and Honkapohja's notation, we cannot otherwise show $\theta = 0$ and $\phi_u = 0$ in the multivariate case.

Now calculate as in Evans and Honkapohja (1986, Proposition 2) to obtain the coefficients for the general ARMA solution to (A.5). Since all solutions to (2.1) also solve (A.5), we need only substitute the general ARMA solution for (A.5) into (2.1) to find the general ARMA solution to (2.1). Define β_n inductively by $\beta_0 = I$ and $\beta_{s+1} = \sum_{r=0}^{s} \pi_r \beta_{s-r}$.[8] We temporarily specialize to the case $n = 1$ to avoid some messy algebra. It is then a straightforward, but tedious, calculation to find the coefficients solving our original system, (2.1).

Proposition 2. *Suppose $q = 1$ and consider the ARMA solutions to (2.1) which are of maximal AR degree under the assumption that $n = 1$. That is*

$$x_t = \Pi(L)x_{t-1} + \Psi(L)v_t + \chi(L)\omega_t \tag{A.6}$$

where ω_t is arbitrary white noise, $\deg \Pi = k + l^* - 1$, $\deg \Psi = k + m^*$, *and* $\deg \chi = k$. *Assuming that (A.6) is a minimal degree AR solution then the coefficients of* Π, Ψ *and* χ *are given by*

$$\pi_s = \begin{cases} -B_k^{-1} B_{k-1-s} & \text{for } s = 0, \ldots, k-2 \\ -B_k^{-1}(B_{k-1-s} + D_{s+1-k}) & \text{for } s = k-1 \\ -B_k^{-1} D_{s+1-k} & \text{for } s = k, \ldots, k + l^* - 1 \end{cases}$$

$$\psi_s = -B_k^{-1} F_{s-k} \qquad \text{for } s = k+1, \ldots, k + m^*,$$

while the other coefficients satisfy

$$-F_0 = -\psi_0 + \sum_{r=0}^{k} \left(\sum_{s=r}^{k} A_{0s} \beta_{s-r} \right) \psi_r \qquad \text{and} \qquad 0 = -\chi_0 + \sum_{r=0}^{k} \left(\sum_{s=r}^{k} A_{0s} \beta_{s-r} \right) \chi_r.$$

[8] A more explicit formula for β_s is given in our 1995 working paper.

REMARK ON n. If $n > 1$ in (2.1) we would get further restrictions on $\psi_0, \dots, \psi_k$ and on the χ's. In this case we subtract $E_{t-n}x_t$ from x_t to obtain

$$0 = \sum_{r=0}^{n} \sum_{s=0}^{k} A_{rs}[E_{t-r}x_{t+s} - E_{t-n}x_{t+s}] + \sum_{s=1}^{n-1} D_s[x_{t-s} - E_{t-n}x_{t-s}] + \sum_{s=0}^{n-1} F_s v_{t-s}.$$

Using Lemma 1, and collecting the various v_{t-s} and ω_{t-s} terms reduces this to a set of $2n$ equations similar to those in Proposition 2.

Notice that if $q > 1$, (A.6) with coefficients as above solves (2.1). However, we cannot conclude it is the only solution of maximal AR degree. When $q = 1$, proposition 4 of Evans and Honkapohja (1986) shows that the solutions we find represent all the solutions of finite ARMA representation while Proposition 3 deals with potential factorizations of the solution. Some of the coefficients $\psi_0, \dots, \psi_k$ and $\chi_0, \dots, \chi_k$ may be arbitrary. By choosing them appropriately, the representation in (A.6) can sometimes be factored. What our theorem tells us is that if there exists a unique nonexplosive solution then the general ARMA solution (A.6) must be explosive since it is not unique. There must, therefore, exist unique values of $\psi_0, \dots, \psi_k$ that allow us to factor (A.6) and arrive at the unique nonexplosive solution, as in equations (3.6) and (3.7).

References

Barro, Robert J. (1989), Interest rate rules, *J. Mon. Econ.* **23**, 3–30.

Blanchard, Olivier J, and Charles M. Kahn (1980), The solutions of linear difference models under rational expectations, *Econometrica* **48**, 1305–1311.

Boyd, John H., III and Michael Dotsey, "Interest Rate Rules and Nominal Determinacy," (long version), Working Paper, University of Rochester, 1995.

Broze, L., C. Gourieroux and A. Szafarz (1985), Solutions of linear rational expectations models, *Econometric Theory* **1**, 341–368.

Canzoneri, Matthew, Dale Henderson and Kenneth S. Rogoff (1983), The information content of the interest rate and optimal monetary policy, *Quart. J. Econ.* **98**, 546–566.

Dotsey, Michael and Robert King (1983), Monetary instruments and policy rules in a rational expectations environment, *J. Mon. Econ.* **12**, 357–382.

Dotsey, Michael and Robert King (1986), Informational implication of interest rate rules, *Amer. Econ. Rev.* **76**, 33–42.

Evans, George W. (1987), The structure of ARMA solutions to a general linear model with rational expectation, *J. Econ. Dyn. Control* **11**, 79–91.

Evans, George W. and Seppo Honkapohja (1986), A complete characterization of ARMA solutions to linear rational expectations models, *Rev. Econ. Studies* **53**, 277–239.

Goodfriend, Marvin (1987), Interest rate smoothing and price level trend stationarity, *J. Mon. Econ.* **19**, 335–348.

Gourieroux, C., Jean-Jaques Laffont and A. Monfort (1982), Rational expectations in dynamic linear models: Analysis of the solutions, *Econometrica* **50**, 490-425.

McCallum, Bennett T. (1981), Price level determinacy with an interest rate policy rule and rational expectations, *J. Mon. Econ.* **8**, 319–329.

McCallum, Bennett T. (1983), On non-uniqueness in rational expectations models: An attempt at perspective, *J. Mon. Econ.* **11**, 139–168.

McCallum, Bennett T. (1986), Some issues concerning interest rate pegging, price-level determinacy, and the real bills doctrine, *J. Mon. Econ.* **17**, 135–160.

Pesaran, M. H. (1981), Identification of rational expectations models, *J. Econometrics* **16**, 375–398.

Pesaran, M. H., "The Limits to Rational Expectations," Basil Blackwell, New York, 1987.

Poole, William W. (1970), Optimal choice of monetary instruments in a simple stochastic growth model, *Quart. J. Econ.* **84**, 197–216.

Sargent, Thomas J., "Macroeconomic Theory," Academic Press, New York, 1979.

Sargent, Thomas J. and Neil Wallace (1975), Rational expectations, the monetary instrument, and the optimal money supply rule, *J. Polit. Econ.* **83**, 241–254.

Whiteman, Charles H., "Linear Rational Expectations Models: A User's Guide," University of Minnesota Press, Minneapolis, 1983.